33

A GENERAL TREATMENT OF PENETRATION FACTOR IN ALPHA-DECAY

S. K. DUTTA

Department of Theoretical Physics,

Indian Association for the Cultivation of Science, Jadavpur, Calcutta-32

(*Received March* 29, 1961)

ABSTRACT. A general treatment of the calculation of the penetration probability of alpha-particles through a potential barrier of Woods-Saxon diffuse type nuclear potential along with the Coulomb potential has been given according to the one-body model for arbitrary values of the angular momentum of the emitted alpha-particles. In the region where practically only the Coulomb potential is present the rigorous solution of Schrödinger equation has been taken from that of Abramowitz. Near the nuclear boundary where both the potentials operate, the Schrödinger equation has been solved by an ingenious method due to Lanczos.

INTRODUCTION

Uptil now, the calculation of the penetration factor in alpha-decay has mostly been based on the WKB method. But it does not seem justified to put much reliance on the results derived by this method without proper investigation specially when the validity of that method has been doubted at times (Blatt and Weisskopf, 1954). So this problem has been tried by a method due to Lanczos (1938). In doing so, the one-body model was followed and for the nuclear potential use was made of the Woods-Saxon diffuse potential which drops exponentially beyond the nuclear surface, besides that the Coulomb potential is present throughout the region. In the region where the Coulomb field predominates the Schrödinger equation has been solved by the Riccati-I method as treated by Abramowitz (1949) (c.f. Froberg, 1955). The Schrödinger equation in the neighbourhood of the nuclear boundary has been solved by the method of Lanczos (1938).

In a previous publication by Dutta, Mitra and Sil (1960), hereafter referred to as I, a calculation of the penetration factor in the process of alpha-decay following one body model has been given for the case of $l = 0$, where the potential field was the same as in this case. In I, for the sake of simplification the Coulomb potential was taken to be constant near the nuclear surface. In the present paper that simplification has been dispensed with and the equation takes into account any arbitrary values of l. Consequently the method of solution for this general case has been different from that of I.

In the method given in I, the differential equation was equated instead of to zero, to an error term proportional to the Tshebysheff's polynomial of a given order. As a result a finite power series solution was obtained, the coefficients of which were easily calculated with the help of a set of recursion relations. In that case the error term vanished at the zero points of the Tshebysheff's polynomial and the error involved in the solution was at the most 1 in 10^8.

In the present case, the differential equation contains the Woods-Saxon exponential term, the Coulomb $\frac{1}{r}$ term and the centrifugal $\frac{l(l+1)}{r^2}$ term and the method of solution, though different from that in I, has also been given by Lanczos (1938). The approximate solution of such a general differential equation which can not be equated to any polynomial, is obtained in the form of a finite power series solution. The coefficients of the power series are evaluated by demanding the vanishing of the differential equation at the zero points of the Tshebysheff's polynomial and thus by solving a set of simultaneous linear equations.

The method given here is a generalisation of the method given in I, to include the cases of differential equation with non-rational coefficients; for the differential equation with rational coefficients this method yields exactly the same coefficients obtained by the method given in I.

The penetration factor has been calculated from the value of the wave function at the point near the nuclear boundary where the potential energy is equal to the kinetic energy of the emitted alpha-particle.

On comparing our results with those obtained by the WKB method it appears that both the sets agree well with each other.

MATHEMATICAL FORMULATION

The equation for u, which is r times the wave function of the radial part of the Schrödinger equation can be written as

$$\frac{d^2u}{dr^2} + \frac{2m}{\hbar^2}\left[E - U(r) - V(r)\right] u = 0, \qquad \ldots (1)$$

where $$U(r) = \frac{2(Z-2)e^2}{r} + \frac{\hbar^2}{2m}\cdot\frac{l(l+1)}{r^2}$$

and $$V(r) = \frac{-V_0}{1+e^{(r-R)/a}}.$$

For convenience of calculation, we neglect $V(r)$ beyond the point r_1 where the nuclear potential drops to $\frac{-V_0}{100}$.

To solve Eq. (1) the space is divided into two regions $r_2 \leqslant r \leqslant r_1$ and $r > r_1$, r_2 being the point where the potential energy is equal to the kinetic energy of the emitted α-particle. In the latter region only the Coulomb potential is of any value.

To solve Eq. (1) in the region $r_2 \leqslant r \leqslant r_1$ Eq. (1) is rewritten by changing the independent variable to $x = r/a$, as follows :

$$\frac{d^2u}{dx^2} + \left[\frac{\lambda^2}{1+\beta e^x} - f(x) - K^2 \right] u = 0, \qquad \ldots \quad (2)$$

where $\lambda^2 = \frac{2ma^2}{\hbar^2} V_0$, $\beta = e^{-R/a}$, $K^2 = \frac{2ma^2}{\hbar^2} \{U(ax_1) - E\}$

and $$f(x) = \frac{2ma^2}{\hbar^2} [U(ax) - U(ax_1)].$$

$U(ax)$ and $U(ax_1)$ being the values of $U(r)$ and $U(r_1)$ respectively on changing the independent variable.

Let us suppose the solution to be of the form,

$$u \sim e^{\mp Kx} . F_{\mp}$$

Thus the Eq. (2) becomes

$$\frac{d^2F_{\mp}}{dx^2} \mp 2K \frac{dF_{\mp}}{dx} + \left\{ \frac{\lambda^2}{1+\beta e^x} - f(x) \right\} F_{\mp} = 0 \qquad \ldots \quad (3)$$

Again, substituting $z = e^{-x}$ and putting $\frac{z+\beta}{z} f(x) = \phi(z)$, the Eq. (3) becomes

$$z(z+\beta) \frac{d^2F_{\mp}}{dz^2} + (z+\beta)(1 \pm 2K) \frac{dF_{\mp}}{dz} + \{\lambda^2 - \phi(z)\} F_{\mp} = 0 \qquad \ldots \quad (4)$$

For later calculations the independent variable occurring as the argument of the Tshebysheff's polynomial has to be normalised such that it varies from zero to one; so we make the transformation

$$p = \frac{z - z_1}{z_2 - z_1}$$

Thus we get from Eq. (4)

$$(p+\mu)(p+\nu) \frac{d^2F_{\mp}}{dp^2} + \delta(p+\nu) \frac{dF_{\mp}}{dp} + \{\lambda^2 - \chi(p)\} F_{\mp} = 0, \qquad \ldots \quad (5)$$

where $\mu = \frac{z_1}{z_2 - z_1}$, $\nu = \frac{z_1 + \beta}{z_2 - z_1}$, $\delta = 1 \pm 2k$, $\chi(p) = \phi(z)$.

For F, a certain polynomial of n th order in p has been assumed which satisfy the above differential equation at the zeros of the Tshebysheff's polynomial of order n.

Let $F = \sum_{i=0} a_i p^i$, where $a_0 = 1$.

and $p_1, p_2, \ldots, p_n$ are the roots of the equation

$$T_n(p) = 0,$$

where $T_n(p)$ is the Tshebysheff's polynomial of order n. Then on substitution of the polynomial $\sum_{i=0}^{n} a_i p^i$ for F in the Eq. (5) we have a set of n simultaneous equations :

$$(p_i+\mu)(p_i+\nu)\left(\frac{d^2F}{dp^2}\right)_{p_i} + \delta(p_i+\nu)\left(\frac{dF}{dp}\right)_{p_i} + \{\lambda^2 - \chi(p_i)\}\, F = 0, \quad \ldots \quad (6)$$

which may be written after rearrangement as

$$a_1 f_1(p_i) + a_2 f_2(p_i) + a_3 f_3(p_i) + \ldots + a_n f_n(p_i) = g(p_i),$$

f_r's and g are known functions of p and are given by

$$f_r(p_i) = r(r-1)p_i^{r-2}(p_i+\mu)(p_i+\nu) + r p_i^{r-1}\delta(p_i+\nu) + p_i^r\{\lambda^2 - \chi(p_i)\}$$

and $$g(p_i) = -\{\lambda^2 - \chi(p_i)\}.$$

This set of n simultaneous linear equations can be written as

$$\begin{pmatrix} f_{11} & f_{21} & f_{31} & \cdots & f_{n1} \\ f_{12} & f_{22} & f_{32} & \cdots & f_{n2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ f_{1n} & f_{2n} & f_{3n} & \cdots & f_{nn} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ \cdot \\ \cdot \\ \cdot \\ g_n \end{pmatrix}$$

where $f_{jk} = f_j(p_k)$ and $g_k = g(p_k)$.

From the above the values of the coefficients of the power series are determined. Therefore the solution of the differential equation is known except for an arbitrary

constant multiplier. The two values of δ give two solutions $F_+(p)$ and $F_-(p)$ corresponding to negative and positive value of δ. Therefore, the solution near the surface of the nucleus ($r_2 < r < r_1$) is

$$u = Ae^{-kx}\, F_-(p) + Be^{+Kx}\, F_+(p). \qquad \ldots (7)$$

Now, in the region ($r > r_1$) only Coulomb potential is effective and the Schrödinger equation takes the form

$$\frac{d^2u}{d\rho^2} + \left\{ 1 - \frac{2\eta}{\rho} - \frac{l(l+1)}{\rho^2} \right\} u = 0, \qquad \ldots (8)$$

where $\rho = \alpha \cdot r = \sqrt{\frac{2mE}{\hbar^2}} \cdot r, \qquad 2\eta = \frac{2m}{\hbar^2} \cdot \frac{2(Z-2)e^2}{\alpha}$.

The Eq. (8) has two solutions : $F_l(\eta, \rho)$ regular at the origin and $G_l(\eta, \rho)$ irregular at the origin, and are defined by their asymptotic behaviour :

$$F_l(\eta, \rho) \sim \sin\theta_l,$$
$$G_l(\eta, \rho) \sim \cos\theta_l,$$

when $\rho \to \infty$.

where $\theta_l = \rho - \eta \log 2\rho - \frac{l}{2}\pi + \sigma_l$ and $\sigma_l = \arg \Gamma(i\eta + l + 1)$.

The boundary condition that at infinity the alpha-particle should behave as a free out-going particle is satisfied by the linear combination $G_l + iF_l$, which represents a pure out-going wave.

When l is an integer, F_{l+1}(or G_{l+1}) can be computed with the help of a set of recurrence relations given by Powell (1947) provided F_l (or G_l) and its first derivative F'_l (or G_l') are known. If y_l stands for either $F_l(\eta, \rho)$ or $G_l(\eta, \rho)$ the recurrence relations satisfied by it are :

$$(l+1)\frac{dy_l}{d\rho} = \left[\frac{(l+1)^2}{\rho} + \eta \right] y_l - [(l+1)^2 + \eta^2]^{\frac{1}{2}} y_{l+1}, \qquad \ldots (9)$$

$$l[(l+1)^2 + \eta^2]^{\frac{1}{2}} y_{l+1} = (2l+1)\left[\eta + \frac{l(l+1)}{\rho} \right] y_l - (l+1)[l^2+\eta^2]^{\frac{1}{2}} y_{l-1}, \qquad (10)$$

$$l\frac{dy_l}{d\rho} = (l^2+\eta^2)^{\frac{1}{2}} y_{l-1} - \left(\frac{l^2}{\rho} + \eta \right) y_l, \qquad (11)$$

If the values of F_0(or G_0) and F'_0(or G'_0) are known by the application of Eq. (9) F_1(or G_1) can be computed and subsequently by the application of Eq. (10) the values of F_l (or G_l) for higher values of l can be obtained. The values of F'_l (or G'_l) are computed with the help of Eq. (11).

Now F_0 and G_0 have different representations in the different ranges defined by the values of ρ and η. In the range where $\rho < 2\eta$, which is the case here,

the representations of F_0 and G_0 are given by Abramowitz, based on Riccati's method as quoted by C. G. Froberg (1955).

$$F_0 = \tfrac{1}{2}e^{\Phi(t,\eta)} \qquad G_0 = e^{\bar{\Psi}(t,\eta)}, \quad \text{where} \quad t = \frac{\rho}{2\eta}.$$

$$\Phi(t,\eta) = 2\eta g_0 + g_1 + (2\eta)^{-1}g_2 + (2\eta)^{-2}g_3 + \ldots$$

$$\bar{\Psi}(t,\eta) = -2\eta g_0 + g_1 - (2\eta)^{-1}g_2 + (2\eta)^{-2}g_3 - \ldots$$

where $g_0, g_1, g_2 \ldots$ etc are given functions of t.

The values of the constants A and B in Eq. (7) are found with the help of the continuity condition of u and du/dr at the point $r = r_1$. The values of F_l and dF_l/dr are found to be negligible in comparison with that of G_l and dG_l/dr at that point. Next, the value of u at $r = r_2$ is calculated from the Eq. (7), from which the penetration factor is determined by following the definition of Blatt and Weisskopf (1954),

$$P = \frac{1}{|u(r_2)|^2}.$$

$u(r_2)$ is given by the following expression :

$$u(r_2) = Ae^{-Kx_2} \quad F\ (1) + Be^{+Kx_2} \quad F_1(1)$$

since at $r = r_2$, $\rho = 1$.

We may write the disintegration constant as

$$\lambda = N.P,$$

where N is the number of times the α-particle hits the barrier wall. If the α-particle moves with a velocity v within the crater of the nucleus of radius R, then $N = v/2R$. We determine v from the condition that the motion of the α-particle within the nucleus of radius R is such that the associated waves form nodes at $r = R$.

If $x_{l,k}$ is the $(k+1)$ th root ($kR = 0$, being the first root) of the equation $J_{l+\frac{1}{2}}(kR) = 0$,

then $$N = \frac{h}{4mR^2} \cdot \frac{x_{l,k}}{\pi}.$$

Now the half-life can be calculated from the expression

$$T = \frac{\log_e 2}{\lambda}$$

where $$\lambda = \frac{h \cdot P}{4mR^2} \cdot \frac{x_{l,k}}{\pi}$$

TABLE I

Element	A mass no.	Z charge no.	R in fermi nuclear radius	m in gm. the reduced mass of the α-particle	l	r_2 in fermi where $P.E. = K.E.$ of the α-particle	E in Mev. the α-particle energy with screening correction	$F_{-}(1)$	$F_{+}(1)$	P	P calculated by WKB method	T in sec.	T in sec. Expt. value of half-life
Rn	218	86	9.3749	6.5223×10^{-24}	0	9.5726	7 162	+.78267	−1.2319	3.555×10^{-19}	4.67×10^{-19}	$.6747 \times 10^{-2}$	1.90×10^{-2}
					2	9.5230	6.564	−.76893	−1.9406	$1\ 220 \times 10^{-21}$	1.62×10^{-21}	107.16×10^{-2}	
Th	230	90	9.5231	6.5286×10^{-24}	0	9.5538	4 718	+.72373	−4.54306	4.937×10^{-33}	6.61×10^{-33}	$.5019 \times 10^{12}$	2.528×10^{12}
					3	9.50198	4.404	−.71005	−6.32904	$.7447 \times 10^{-35}$	$1\ 04 \times 10^{-35}$	149.456×10^{12}	
Pu	238	94	9.6190	6.5325×10^{-24}	0	9.64396	5.535	+.725805	−4.7648	7.543×10^{-30}	9.30×10^{-30}	$.335 \times 10^{9}$	2.822×10^{9}
					4	9.5777	5.394	+.711196	−7.13057	1.911×10^{-31}	2.45×10^{-31}	4.994×10^{9}	

TABLE II

Element	A	Z	l	E in MeV.	m in gm.	Value of P by the method of calculation: As given in I	Value of P by the method of calculation: As given in this paper	P by WKB method	Value of T calculated from that of P obtained by using the method: given in I in sec.	Value of T calculated from that of P obtained by using the method: given in this paper in sec.	Expt. value of T in sec.
Po	214	84	0	7.714	6.52×10^{-24}	1.337×10^{-16}	1.179×10^{-16}	1.58×10^{-16}	$.177 \times 10^{-4}$	$.2012 \times 10^{-4}$	1.636×10^{-4}

RESULTS

The numerical calculations have been done for the following elements, for different values of l, to show the applicability of the method. The values of parameters are the same as used by Igo and Thaler (1957).

$$R = 1.35A^{1/3} + 1.3 \text{ fermi}, \quad a = 0.5 \text{ fermi}, \quad V_0 = 45\text{Mev}.$$

In our calculation we have taken $n = 4$ and $k = 1$. Results are shown in Table I.

The values of half-life as calculated here are lower than the experimental values of the same for the case $l = 0$ only, whereas for $l = 2, 3$, or 4 they are all greater than the experimental values. The values of the penetration factor calculated by the method given here appear to be consistently lower than that calculated by the WKB method for all values of l. The values of P by WKB method and the experimental value of half-life and that of E are taken from the table given by Rassmussen (1959) in his paper on the penetration probability of alpha-particles.

It may be worth while to compare the results for the case $l = 0$ as obtained here with that of the previous method where the Coulomb potential near the nuclear boundary was taken to be of constant value. In our present case, as expected, the value of P is slightly lower than that obtained by the previous method. The differences in the figures given in Table II indicate the measure of error involved in the approximation about the Coulomb potential in the previous method.

In the paper 1, due te some numerical slip in the value of a parameter, the penetration factor of $^{214}Po_{84}$ for $l = 0$ was found to be $.059 \times 10^{-16}$ which should, instead, be 1.3373×10^{-16}.

ACKNOWLEDGMENTS

The author is indebted to Prof. D. Basu, Ph. D., for suggesting the problem and helpful guidance throughout the progress of the work and he also wishes to thank Dr. N. C. Sil for many valuable discussions.

REFERENCES

Abramowitz, M., 1949, *Quart. Appl. Math.*, **7**, 75.

Blatt and Weisskopf, 1954, Theoretical Nuclear Physics, p. 332 & 362, John Wiley & Sons, N.Y.

Dutta, Mitra and Sil, 1960, *Ind. J. Phys.*, **34**, 205.

Froberg, C. E., 1955, *Rev. Mod. Phys.*, **27**, 405.

Igo, G. and Thaler, R. M., 1957, *Phys., Rev.*, **106**, 126.

Lanczos, C., 1938, *Jour. Math. Phys.*, **17**, 123.

Powell, J. L., 1947, *Phys. Rev.*, **72**, 626.

Rassmussen, J. O., 1959, *Phys. Rev.*, **113**, 1593.